Бернулијева диференцијална једначина

$y' + P(x)y = Q(x)\cdot y^{\alpha}, \ \alpha \in R$ (*)

За $\alpha = 0$ добијамо линеарну ДЈ (нехомогена)

За $\alpha = 1$ добијамо линеарну ДЈ (хомогена)

За $\alpha \neq 0,1$ уводимо смјену:

$z = y^{1-\alpha}$

$z' = (1-\alpha)y^{1-\alpha-1}\cdot y'$

$z' = (1-\alpha)y'\cdot y^{-\alpha} = \frac{y'}{y^{\alpha}}(1-\alpha)$. (**)

Подијелимо ј-ну (*) са y^{α}:

$\frac{y'}{y^{\alpha}} + P(x)\cdot\frac{y}{y^{\alpha}} = Q(x)\cdot\frac{y^{\alpha}}{y^{\alpha}}$

$\frac{y'}{y^{\alpha}} + P(x)\cdot\frac{1}{y^{\alpha-1}} = Q(x)$ (***)

Из (**) је $\frac{y'}{y^{\alpha}} = \frac{z'}{1-\alpha}$. Пошто је $\frac{1}{y^{\alpha-1}} = y^{1-\alpha} = z$, уврштавањем у (***) добијамо:

$\frac{z'}{1-\alpha} + P(x)\cdot z = Q(x) \quad /(1-\alpha)$

$z' + \underbrace{(1-\alpha)P(x)}_{P_1(x)} z = \underbrace{(1-\alpha)Q(x)}_{Q_1(x)}$.

Добијена једначина је линеарна диференцијална ј-на

Рјешење дате дифенцијалне ј-не је:

$z = e^{-\int P_1(x)\,dx}\left(\int Q_1(x)e^{\int P_1(x)dx}\,dx + C\right)$

1 Ријешити ДЈ.

$xy' - 4y = x^2\sqrt{y} \quad /:x$

$y' - 4\cdot\frac{y}{x} = x\sqrt{y}$

$y' + \left(-\frac{4}{x}\right)y = x\cdot y^{\frac{1}{2}}$

Подијелимо ј-ну са $y^{\frac{1}{2}}$:

$$\frac{y'}{\sqrt{y}}+\left(-\frac{4}{x}\right)\sqrt{y}=x \quad (*)$$

Уводимо смјену: $z=y^{1-\frac{1}{2}}$

$z=y^{\frac{1}{2}}$

$z'=\frac{1}{2}y^{-\frac{1}{2}}\cdot y'=\frac{y'}{\sqrt{y}\cdot 2}$. Одавде: $\frac{y'}{\sqrt{y}}=2z'$, $z=\sqrt{y}$.

Уврштавањем у $(*)$ добијамо:

$$2z'+\left(-\frac{4}{x}\right)z=x \quad /:2$$

$$z'+\underbrace{\left(-\frac{2}{x}\right)}_{P(x)}z=\underbrace{\left(\frac{x}{2}\right)}_{Q(x)}.$$

$$z=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+C\right)$$

$$\int P(x)dx=\int -\frac{2}{x}dx=-2\ln|x|$$

$$z=e^{2\ln|x|}\left(\int \frac{x}{2}e^{-2\ln|x|}dx+C\right)$$

$$e^{2\ln|x|}=e^{\ln|x|^2}=|x|^2=x^2$$

$$e^{-2\ln|x|}=e^{\ln\frac{1}{|x|^2}}=\frac{1}{x^2}$$

$$z=x^2\left(\int \frac{x}{2}\cdot\frac{1}{x^2}dx+C\right)$$

$$z=x^2\cdot\left(\frac{1}{2}\ln|x|+C\right)$$

Пошто је $z=\sqrt{y}$, то:

$$\sqrt{y}=x^2\left(\frac{\ln|x|}{2}+C\right)=x^2\left(\ln\sqrt{|x|}+C\right)$$

$$y=\left(x^2\left(\ln\sqrt{|x|}+C\right)\right)^2$$

② $y'-y=xy^2$, $y(0)=2$.

$$y'+(-1)y=xy^2 \quad /:y^2$$

$$\frac{y'}{y^2}+(-1)\cdot\frac{1}{y}=x \quad (*)$$

Уведимо смјену:

$$z = y^{1-2} = y^{-1}$$

$$z' = -1 \cdot y^{-2} \cdot y' = -\frac{y'}{y^2}$$

Одавде је $\frac{y'}{y^2} = -z'$.

Враћањем у (*),

$$-z' + (-1) \cdot z = x \quad (-1)$$

$$z' + z = -x$$

$$P(x) = 1, \quad Q(x) = -x$$

Сада,

$$z = e^{-\int P(x)dx}\left(\int Q(x) e^{\int P(x)dx} dx + C\right)$$

$$\int P(x)dx = \int dx = x$$

$$\int -x \cdot e^{\int dx} dx = \int -x \cdot e^x dx$$

Даље,

$$z = e^{-x}\left(\int -x e^x dx + C\right) = e^{-x}\left(-\int x e^x dx + C\right)$$

Нађимо $I = \int x e^x dx$.

Парцијална интеграција:

$$\begin{array}{ll} u = x & e^x dx = dv \\ du = dx & v = \int e^x dx = e^x \end{array}$$

Сада, $I = x \cdot e^x - \int e^x dx = x e^x - e^x$.

Рјешење:

$$z = e^{-x}(-x e^x + e^x + C) = -x + 1 + C \cdot e^{-x}$$

Због $z = \frac{1}{y}$, одавде $y = \frac{1}{z}$, тј.

$$y = \frac{1}{Ce^{-x} - x + 1}.$$

Због почетног услова, $y(0) = 2$ добијамо:

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$$2=\frac{1}{c\cdot e^{0}-0+1}$$

$$2=\frac{1}{c+1}$$

$$c+1=\frac{1}{2}$$

$$\boxed{c=-\frac{1}{2}}$$

Рјешење: $\boxed{y=\frac{1}{-\frac{1}{2}e^{-x}-x+1}}$

3 $y'+2xy=2x^3y^3$

Подијелимо са y^3.

$$\frac{y'}{y^3}+\frac{2x}{y^2}=2x^3. \quad (*)$$

Смјена: $z=y^{1-3}=y^{-2}$

$$z'=-2y^{-3}\cdot y'$$

$$z'=-2\cdot\frac{y'}{y^3}.$$

Одавде $\frac{y'}{y^3}=\frac{z'}{-2}$

Враћањем у $(*)$,

$$\frac{z'}{-2}+2x\cdot z=2x^3 \quad /(-2)$$

$$z'-4x\cdot z=-4x^3$$

$$P(x)=-4x, \ Q(x)=-4x^3$$

$$z=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+C\right)$$

$$\int P(x)dx=\int -4x\,dx=-\frac{4x^2}{2}=-2x^2$$

$$z=e^{2x^2}\left(\int -4x^3\cdot e^{-2x^2}dx+C\right)$$

$$=e^{2x^2}\left(-4\int x^3\cdot e^{-2x^2}dx+C\right)$$

Нађимо $I=\int x^3 \cdot e^{-2x^2} dx$.

Метод парцијалне интеграције:

$$\begin{array}{ll} u=x^2 & x\cdot e^{-2x^2}dx = dv \\ du=2xdx & v=\int x\cdot e^{-2x^2}dx = \left[\begin{array}{l} -2x^2=t \\ -4xdx=dt \end{array} \right] = \int \frac{x\cdot e^{t}\cdot dt}{-4x} = \\ & =-\frac{1}{4}e^{t}=-\frac{1}{4}e^{-2x^2} \end{array}$$

Сада, $I=-\frac{1}{4}x^2e^{-2x^2}-\int -\frac{1}{4}e^{-2x^2}2\cdot x\,dx$

$$=-\frac{1}{4}x^2e^{-2x^2}+\frac{2}{4}\int x\cdot e^{-2x^2}dx$$

$$=-\frac{1}{4}x^2e^{-2x^2}+\frac{1}{2}\left(-\frac{1}{4}\cdot e^{-2x^2}\right)=$$

$$=-\frac{1}{4}x^2\cdot e^{-2x^2}-\frac{1}{8}e^{-2x^2}$$

Добијамо,

$$z=e^{2x^2}\left(-4\left(-\frac{1}{4}x^2e^{-2x^2}-\frac{1}{8}e^{-2x^2}\right)+C\right)$$

$$=e^{2x^2}\left(x^2\cdot e^{-2x^2}+\frac{1}{2}e^{-2x^2}+C\right)$$

$$=x^2+\frac{1}{2}+C\cdot e^{2x^2}$$

Пошто је $z=\frac{1}{y^2}$, одавде је $y^2=\frac{1}{x^2+\frac{1}{2}+C\cdot e^{2x^2}}$, па је:

$$y=\pm\sqrt{\frac{1}{x^2+\frac{1}{2}+C\cdot e^{2x^2}}}$$

4) $(x^2+y^2+2x)dx+2ydy=0, \quad y(0)=1$

$$(x^2+y^2+2x)dx=-2y\,dy$$

$$\frac{dy}{dx}=\frac{x^2+y^2+2x}{-2y}$$

$$y'=-\frac{y}{2}+\frac{x^2+2x}{-2y}$$

$$y'+\frac{y}{2}=\frac{1}{y}\cdot\left(\frac{x^2+2x}{-2}\right)$$

$$y' + \frac{y}{2} = \frac{1}{y} \cdot \frac{x^2+2x}{-2} = \frac{(x^2+2x)}{-2} \cdot y^{-1}.$$

$\alpha = -1.$

Подијелимо са y^{-1}:

$$y' \cdot y + \frac{1}{2} y^2 = -\frac{x^2-2x}{2}.$$

Смјена: $z = y^{1-(-1)} = y^2$.

$z' = 2 \cdot y \cdot y'$, па:

$y' \cdot y = \frac{z'}{2}$

$$\frac{z'}{2} + \frac{1}{2} z = -\frac{x^2-2x}{2} \quad /2$$

$$z' + z = -x^2 - 2x \quad \text{(линеарна ДЈ)}$$

$P(x) = 1, \quad Q(x) = -x^2-2x$

$$z = e^{-\int P(x)dx} \left(\int Q(x) e^{\int P(x)dx} dx + C \right)$$

$$\int P(x)dx = \int dx = x$$

$$z = e^{-x} \left(\int (-x^2-2x) \cdot e^x dx + C \right) =$$

$$e^{-x} \left(-\int (x^2+2x) e^x dx + C \right)$$

$$I = \int (x^2+2x) e^x dx = \begin{bmatrix} u = x^2+2x & e^x dx = dv \\ du = (2x+2)dx & v = \int e^x dx = e^x \end{bmatrix}$$

$$= (x^2+2x) \cdot e^x - \underbrace{\int e^x (2x+2) dx}_{= I_1} = (x^2+2x)e^x - (2x+2)e^x + 2e^x$$

$$I_1: \begin{bmatrix} u = 2x+2 & e^x dx = dv \\ du = 2dx & v = \int e^x dx = e^x \end{bmatrix}$$

$$I_1 = (2x+2)e^x - \int 2e^x dx = (2x+2)e^x - 2e^x$$

$$z=e^{-x}\left(-(x^2+2x)e^x+(2x+2)e^x-2e^x+C\right)$$

$$z=e^{-x}\left(e^x(-x^2-\cancel{2x}+\cancel{2x}+\cancel{2}-\cancel{2})+C\right)$$

$$=e^{-x}\left(e^x(-x^2)+C\right).$$ Због $z=y^2$

$$\underbrace{y^2=-x^2+e^{-x}\cdot C}$$

Уз услов $y(0)=1$,

$$1^2=-0+e^0\cdot C$$

$$\underbrace{1=C}$$

$y^2=-x^2+e^{-x}$, па $\underbrace{y=\pm\sqrt{e^{-x}-x^2}}$

⑤ $y'-2y\cdot tg\,x+y^2 tg^4x=0$, $y(0)=2$

$y'+y(-2tg\,x)=-tg^4x\cdot y^2 \quad (\alpha=2)$

Подијелимо ј-ну са y^2.

$$\frac{y'}{y^2}+\frac{(-2tg\,x)}{y}=-tg^4x$$

Смјена: $z=y^{1-2}=y^{-1}$

$$z'=-1y^{-2}\cdot y'$$

$z'=-\frac{y'}{y^2}$. Одавде $\frac{y'}{y^2}=-z'$

$$-z'+(-2tg\,x)\cdot\frac{1}{y}=-tg^4x \quad /(-1)$$

$$z'+\underbrace{(2tg\,x)}_{P(x)}\underbrace{\frac{1}{y}}_{z}=\underbrace{(tg^4x)}_{Q(x)}$$

$$z=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+C\right)$$

$$\int P(x)dx=\int 2tg\,x\,dx=2\int\frac{\sin x}{\cos x}dx=\begin{bmatrix}\cos x=t\\ -\sin x\,dx=dt\end{bmatrix}=2\cdot\int-\frac{dt}{t}=-2\ln|\cos x|$$

$$Z = e^{2\ln|\cos x|}\left(\int tg^4 x \cdot e^{-2\ln|\cos x|} \cdot dx + C\right)$$

$$Z = e^{\ln(\cos x)^2}\left(\int tg^4 x \cdot e^{\ln\frac{1}{\cos^2 x}} \cdot dx + C\right)$$

$$Z = \cos^2 x \cdot \left(\int tg^4 x \cdot \frac{1}{\cos^2 x} \cdot dx + C\right)$$

$$Z = \cos^2 x \cdot \left(\int \frac{tg^4 x}{\cos^2 x} dx + C\right)$$

$$\int \frac{tg^4 x}{\cos^2 x} dx = \begin{bmatrix} tg\, x = t \\ \frac{dx}{\cos^2 x} = dt \end{bmatrix} = \int t^4 \cdot dt = \frac{t^5}{5} = \frac{tg^5 x}{5}$$

Добијамо,

$$Z = \cos^2 x \cdot \left(\frac{tg^5 x}{5} + C\right)$$

$$\frac{1}{y} = \cos^2 x\left(\frac{tg^5 x}{5} + C\right) = \cos^2 x\left(\frac{\sin^5 x}{5\cos^5 x} + C\right)$$

$$y = \frac{1}{\cos^2 x\left(\frac{\sin^5 x + 5C \cdot \cos^5 x}{5\cos^5 x}\right)} = \frac{5\cos^3 x}{\sin^5 x + C_1 \cdot \cos^5 x}$$

$$y(0) = 2 \Rightarrow 2 = \frac{5 \cdot \cos^3 0}{\sin^5 0 + C_1 \cdot \cos^5 0}$$

$$2 = \frac{5}{0 + C_1} \Rightarrow C_1 = \frac{5}{2}$$

Сада, $y = \dfrac{5\cos^3 x}{\sin^5 x + \frac{5\cos^5 x}{2}} = \dfrac{10\cos^3 x}{2\sin^5 x + 5\cos^5 x}$

Диференцијалне ј-не са тоталним диференцијалом ①

Ако је задата ј-на:

$$P(x,y)\,dx + \Theta(x,y)\,dy = 0$$ и притом $\frac{\partial P}{\partial y} = \frac{\partial \Theta}{\partial x}$, онда се она назива диференцијалном ј-ном с тоталним диференцијалом.

Лијева страна је тада тотални диференцијал неке ф-је $u=u(x,y)$.

Дакле, тражимо u, тд. $du = P\,dx + \Theta\,dy$, $P = \frac{\partial u}{\partial x}$, $\Theta = \frac{\partial u}{\partial y}$

$du = 0 \rightarrow u(x,y) = C$.

Пошто $\frac{\partial u}{\partial x} = P$, то је $u = \int P\,dx + \varphi(y)$.

Диференцирањем по y,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int P\,dx + \varphi(y)\right) = \frac{\partial}{\partial y}\int P\,dx + \frac{\partial \varphi}{\partial y}$$

Пошто $\frac{\partial u}{\partial y} = \Theta$, онда:

$$\Theta = \frac{\partial}{\partial u}\int P(x)\,dx + \frac{\partial \varphi}{\partial y}$$

Одавде, $\frac{\partial \varphi}{\partial y} = \Theta - \frac{\partial}{\partial u}\int P(x)\,dx$, тј. $\varphi = \int\left(\Theta - \frac{\partial}{\partial u}\int P(x)\,dx\right)dy$.

Рјешење:

$$u(x,y) = \int P\,dx + \int\left(\Theta - \frac{\partial}{\partial y}\int P\,dx\right)dy = C$$

① $\frac{2x}{y^3}\,dx + \frac{y^2-3x^2}{y^4}\,dy = 0$

$\frac{\partial P}{\partial y} = -6xy^{-4} = -\frac{6x}{y^4}$ $\Rightarrow$ ј-на са тоталним диференцијалом

$\frac{\partial \Theta}{\partial x} = -\frac{6x}{y^4}$

Тражимо ф-ју u, тд.

$u = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, гдје је $\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = \Theta$

$$\frac{\partial u}{\partial x} = \frac{2x}{y^3}$$

$$u = \int \frac{2x}{y^3}\,dx + \varphi(y) = \frac{x^2}{y^3} + \varphi(y)$$

$$\frac{\partial u}{\partial y} = x^2(-3y^{-4}) + \varphi'(y) = \frac{-3x^2}{y^4} + \varphi'(y)$$

$$\Theta = \frac{y^2-3x^2}{y^4} = -\frac{3x^2}{y^4} + \varphi'(y)$$

$$\frac{1}{y^2} - \cancel{\frac{3x^2}{y^4}} = -\cancel{\frac{3x^2}{y^4}} + \varphi'(y)$$

$$\varphi'(y) = \frac{1}{y^2}$$

$$\varphi(y) = \int \frac{dy}{y^2} = -\frac{1}{y}$$

Сада, $u(x,y) = \underline{\frac{x^2}{y^3} - \frac{1}{y} = C}$,

② $\underbrace{(3x^2+6xy^2)}_{P}dx + \underbrace{(6x^2y+4y^2)}_{Q}dy = 0,\ y(1)=2$

$$\frac{\partial P}{\partial y} = 12xy$$
$$\frac{\partial Q}{\partial x} = 12xy$$
$\Rightarrow$ j-на с тоталним диференцијалом

Тражимо u, тд.

$$\frac{\partial u}{\partial x} = P,\ \frac{\partial u}{\partial y} = Q,\ \text{тј. да}\ du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}\cdot Q$$

$$\frac{\partial u}{\partial x} = 3x^2+6xy^2$$

$$u = \int (3x^2+6xy^2)dx + \varphi(y) =$$
$$= \frac{3x^3}{3} + 6y^2\cdot\frac{x^2}{2} + \varphi(y)$$
$$= x^3 + 3x^2y^2 + \varphi(y)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3+3x^2y^2+\varphi(y)) = 6x^2y + \varphi'(y)$$

$Q = 6x^2y+4y^2 = \frac{\partial u}{\partial y}$, тј.

$$6\cancel{x^2y} + 4y^2 = 6\cancel{x^2y} + \varphi'(y)$$

$$\varphi'(y) = 4y^2$$

$$\varphi(y) = \int 4y^2 dy = \frac{4y^3}{3}$$

Сада, $u(x,y) = x^3 + 3x^2y^2 + \frac{4y^3}{3} = C$

③ $\left(\frac{y}{x+y}\right)^2 dx + \left(\frac{x}{x+y}\right)^2 dy = 0$

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left(\frac{y^2}{(x+y)^2}\right)=\frac{2y(x+y)^2-y^2\cdot 2(x+y)}{(x+y)^4}=\frac{2xy\cancel{(x+y)}}{(x+y)^{\cancel{4}\,3}}=\frac{2xy}{(x+y)^3}$$

$$\frac{\partial Q}{\partial x}=\frac{2x(x+y)^2-x^2\cdot 2(x+y)}{(x+y)^4}=\frac{2x\cancel{(x+y)}(\cancel{x}+y-\cancel{x})}{(x+y)^{\cancel{4}\,3}}=\frac{2xy}{(x+y)^3}$$

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\Rightarrow$ ј-на са тоталним диференц.

$u=?$

$$u=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=\frac{y^2}{(x+y)^2}\Rightarrow u=\int\frac{y^2}{(x+y)^2}dx+\varphi(y)$$

$$\left[\begin{array}{l}x+y=t\\ dx=dt\end{array}\right.$$

$$\int\frac{y^2}{(x+y)^2}dx=\int\frac{y^2dt}{t^2}=y^2\cdot\left(-\frac{1}{t}\right)=y^2\cdot\left(\frac{-1}{x+y}\right)=\frac{-y^2}{x+y}$$

$$u=\frac{-y^2}{x+y}+\varphi(y)$$

$$\frac{\partial u}{\partial y}=-\left(\frac{2y(x+y)-y^2\cdot 1}{(x+y)^2}\right)+\varphi'(y)=-\left(\frac{2xy+y^2}{(x+y)^2}\right)+\varphi'(y)$$

Из $\frac{\partial u}{\partial y}=Q$, $-\frac{2xy+y^2}{(x+y)^2}+\varphi'(y)=\frac{x^2}{(x+y)^2}$, па:

$$\varphi'(y)=\frac{x^2+2xy+y^2}{(x+y)^2}=1$$

$$\varphi'(y)=1\Rightarrow\varphi(y)=\int dy=y$$

$$u(x,y)=\frac{-y^2}{(x+y)}+y=C$$

$$\frac{-y^2+xy+y^2}{x+y}=C$$

$$\boxed{\frac{xy}{x+y}=C}$$

4) $$\underbrace{\left(\frac{x}{\sin y}+2\right)}_{P}dx+\underbrace{\left(\frac{(x^2+1)\cos y}{\cos 2y-1}\right)}_{Q}dy=0$$

$$P=\frac{x}{\sin y}+2,\quad Q=\frac{(x^2+1)\cos y}{\cos 2y-1}$$

$$\frac{\partial P}{\partial y} = -\frac{x}{\sin^2 y}\cdot\cos y \qquad \frac{\partial Q}{\partial x} = \frac{2x\cos y}{\cos 2y - 1} = \frac{2x\cos y}{-2\sin^2 y}$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$ ј-на с тоталним диференцијалом

$\ulcorner 1-\cos 2y = 2\sin^2 y \lrcorner$

Тражимо u, $u = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$\frac{\partial u}{\partial x} = \frac{x}{\sin y} + 2$$

$$u = \int\left(\frac{x}{\sin y} + 2\right)dx + \varphi(y)$$

$$= \frac{x^2}{2\sin y} + 2x + \varphi(y)$$

$$\frac{\partial u}{\partial y} = \frac{x^2}{2}\cdot\frac{(-1)}{\sin^2 y}\cdot\cos y + \varphi'(y) = -\frac{x^2\cos y}{2\sin^2 y} + \varphi'(y)$$

$$-\frac{x^2\cos y}{2\sin^2 y} + \varphi'(y) = \frac{(x^2+1)\cos y}{\cos 2y - 1}$$

$$\varphi'(y) = \frac{-x^2\cos y - \cos y}{2\sin^2 y} + \frac{x^2\cos y}{2\sin^2 y} = \frac{-\cos y}{2\sin^2 y}$$

$$\varphi(y) = \int\frac{-\cos y}{2\sin^2 y}dy = \begin{bmatrix}\sin y = t\\ \cos y\,dy = dt\end{bmatrix} = \int -\frac{dt}{2t^2} = -\frac{1}{2}\cdot\left(-\frac{1}{t}\right) = \frac{1}{2t} =$$

$$= \frac{1}{2\sin y}$$

$$u = \frac{x^2}{2\sin y} + 2x + \frac{1}{2\sin y} = C$$

$$\frac{x^2+1}{2\sin y} + 2x = C$$

5) $\underbrace{x(1+y^2)}_{P}dx + \underbrace{\left(x^2y + \frac{1}{\sqrt{1-y^2}}\right)}_{Q}dy = 0$

$$\frac{\partial P}{\partial y} = x\cdot 2y = 2xy$$

$\Rightarrow$ ј-на с тоталним диференцијалом.

$$\frac{\partial Q}{\partial x} = 2xy$$

Пражимо u, що

$$u = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$u = \int x(1+y^2)dx = (1+y^2)\cdot\frac{x^2}{2} + f(y)$$

$$\frac{\partial u}{\partial y} = \frac{x^2}{2}\cdot 2y + f'(y) = x^2y + f'(y).$$

$$\cancel{x^2y} + f'(y) = \cancel{x^2y} + \frac{1}{\sqrt{1-y^2}}$$

$$f(y) = \int \frac{dy}{\sqrt{1-y^2}} = \arcsin y$$

Далі,

$$u = (1+y^2)\frac{x^2}{2} + \arcsin y = C$$

$$\boxed{(1+y^2)\frac{x^2}{2} + \arcsin y = C}$$